Math 150, ***Lecture Notes- Bonds*** **Name____________**

Section 2.4 *The Chain Rule*

THEOREM 2.10 The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

PROOF Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c. A problem occurs if there are values of x, other than c, such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c. In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \to g(c)$ as $x \to c$.

$$\begin{aligned} h'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\ &= \lim_{x \to c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \right], \quad g(x) \neq g(c) \\ &= \left[\lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right] \left[\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right] \\ &= f'(g(c))g'(c) \end{aligned}$$

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When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.

Outer function

$$y = f(g(x)) = f(u)$$

Inner function

The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) *times* the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

Ex.1 Writing the decomposition of a composite function.

	$y = f(g(x))$	$u = g(x)$	$y = f(u)$
a.	$y = \dfrac{1}{x+1}$	$u = x + 1$	$y = \dfrac{1}{u}$
b.	$y = \sin 2x$	$u = 2x$	$y = \sin u$
c.	$y = \sqrt{3x^2 - x + 1}$	$u = 3x^2 - x + 1$	$y = \sqrt{u}$
d.	$y = \tan^2 x$	$u = \tan x$	$y = u^2$

Ex.2 Find the derivative of $y = 5\left(2 - x^3\right)^4$.

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = n u^{n-1} u'.$$

PROOF Because $y = u^n$, you apply the Chain Rule to obtain

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = \frac{d}{du}[u^n]\frac{du}{dx}.$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = n u^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1}\frac{du}{dx}.$$

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Ex.3 Find the derivative of $g(t)=8\sqrt[4]{9-t^2}$.

Summary of Differentiation Rules

General Differentiation Rules	Let f, g, and u be differentiable functions of x.		
	Constant Multiple Rule: $\frac{d}{dx}[cf] = cf'$	*Sum or Difference Rule*: $\frac{d}{dx}[f \pm g] = f' \pm g'$	
	Product Rule: $\frac{d}{dx}[fg] = fg' + gf'$	*Quotient Rule*: $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$	
Derivatives of Algebraic Functions	*Constant Rule*: $\frac{d}{dx}[c] = 0$	*(Simple) Power Rule*: $\frac{d}{dx}[x^n] = nx^{n-1}$, $\frac{d}{dx}[x] = 1$	
Derivatives of Trigonometric Functions	$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\tan x] = \sec^2 x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$
	$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$
Chain Rule	*Chain Rule*: $\frac{d}{dx}[f(u)] = f'(u)\, u'$	*General Power Rule*: $\frac{d}{dx}[u^n] = nu^{n-1}u'$	

Ex.4 Find the derivative of $y=x^2\sqrt{16-x^2}$.

Ex.5 Find on the graph of $f(x)=\sqrt[3]{\left(x^2-1\right)^2}$ for which $f'(x)=0$ and those for which $f'(x)$ does not exist.

Ex.6 Find the derivative of $y=\dfrac{t}{\sqrt{t^4+4}}$.

Ex.7 Find the derivative of $h(t)=\left(\dfrac{t^2}{t^3+2}\right)^2$.

Trigonometric Functions and the Chain Rule

The "Chain Rule versions" of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u)\,u' \qquad \frac{d}{dx}[\cos u] = -(\sin u)\,u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\,u' \qquad \frac{d}{dx}[\cot u] = -(\csc^2 u)\,u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u)\,u' \qquad \frac{d}{dx}[\csc u] = -(\csc u \cot u)\,u'$$

Ex.8 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \overbrace{2x}^{u}$ $\qquad y' = \overbrace{\cos 2x}^{\cos u}\,\overbrace{\frac{d}{dx}[2x]}^{u'} = (\cos 2x)(2) = 2\cos 2x$

b. $y = \cos(x-1)$ $\qquad y' = -\sin(x-1)$

c. $y = \tan 3x$ $\qquad y' = 3\sec^2 3x$

Ex.9 Derivatives, Parentheses, and Trigonometric Functions

Find the derivative of the following functions:

(a) $y = \cos 3x^2 = \cos(3x^2)$

(b) $y = (\cos 3)x^2$

(c) $y = \cos(3x)^2 = \cos(9x^2)$

(d) $y = \cos^2 x = (\cos x)^2$

Ex.10 Find the derivative of $g(\theta) = 5\cos^2(\pi\theta)$.

Ex.11 Find the derivative of $g(\theta)=\cos\sqrt{\sin(\tan(\pi\theta))}$.

Ex.12 Evaluate the second derivative of $g(\theta)=\tan(2\theta)$ at $\left(\frac{\pi}{6},\sqrt{3}\right)$.

Ex.13 Find the equation of the tangent line to the graph of $f(x) = 2\sin(x) + \cos(2x)$ at $(\pi, 1)$. Then, find all values of x in $(0, 2\pi)$ at which the graph of f has a horizontal tangent.

Ex.14 Given $h(x) = f(g(x))$ and $s(x) = g(f(x))$, use the graphs of f and g to find the following derivatives:

(a) Find $h'(3)$.

(b) Find $s'(9)$.