

Section 2.4 The Chain Rule**THEOREM 2.10 The Chain Rule**

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x , then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$$

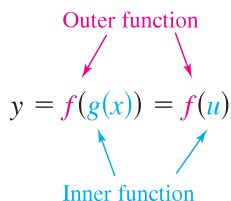
PROOF Let $h(x) = f(g(x))$. Then, using the alternative form of the derivative, you need to show that, for $x = c$,

$$h'(c) = f'(g(c))g'(c).$$

An important consideration in this proof is the behavior of g as x approaches c . A problem occurs if there are values of x , other than c , such that $g(x) = g(c)$. Appendix A shows how to use the differentiability of f and g to overcome this problem. For now, assume that $g(x) \neq g(c)$ for values of x other than c . In the proofs of the Product Rule and the Quotient Rule, the same quantity was added and subtracted to obtain the desired form. This proof uses a similar technique—multiplying and dividing by the same (nonzero) quantity. Note that because g is differentiable, it is also continuous, and it follows that $g(x) \rightarrow g(c)$ as $x \rightarrow c$.

$$\begin{aligned} h'(c) &= \lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{x - c} \\ &= \lim_{x \rightarrow c} \left[\frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \cdot \frac{g(x) - g(c)}{x - c} \right], \quad g(x) \neq g(c) \\ &= \left[\lim_{x \rightarrow c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \right] \left[\lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \right] \\ &= f'(g(c))g'(c) \quad \blacksquare \end{aligned}$$

When applying the Chain Rule, it is helpful to think of the composite function $f \circ g$ as having two parts—an inner part and an outer part.



The derivative of $y = f(u)$ is the derivative of the outer function (at the inner function u) times the derivative of the inner function.

$$y' = f'(u) \cdot u'$$

Ex.1 Writing the decomposition of a composite function.

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$

a. $y = \frac{1}{x+1}$

$$u = x + 1$$

$$y = \frac{1}{u}$$

b. $y = \sin 2x$

$$u = 2x$$

$$y = \sin u$$

c. $y = \sqrt{3x^2 - x + 1}$

$$u = 3x^2 - x + 1$$

$$y = \sqrt{u}$$

d. $y = \tan^2 x$

$$u = \tan x$$

$$y = u^2$$

Ex.2 Find the derivative of $y = 5(2 - x^3)^4$.

THEOREM 2.11 The General Power Rule

If $y = [u(x)]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}[u^n] = nu^{n-1} u'$$

PROOF Because $y = u^n$, you apply the Chain Rule to obtain

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) \\ &= \frac{d}{du}[u^n] \frac{du}{dx}. \end{aligned}$$

By the (Simple) Power Rule in Section 2.2, you have $D_u[u^n] = nu^{n-1}$, and it follows that

$$\frac{dy}{dx} = n[u(x)]^{n-1} \frac{du}{dx}.$$



Ex.3 Find the derivative of $g(t) = 8\sqrt[4]{9-t^2}$.

Summary of Differentiation Rules

General Differentiation Rules

Let f , g , and u be differentiable functions of x .

Constant Multiple Rule:

$$\frac{d}{dx}[cf] = cf'$$

Product Rule:

$$\frac{d}{dx}[fg] = fg' + gf'$$

Constant Rule:

$$\frac{d}{dx}[c] = 0$$

Derivatives of Algebraic Functions

Sum or Difference Rule:

$$\frac{d}{dx}[f \pm g] = f' \pm g'$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{gf' - fg'}{g^2}$$

(Simple) Power Rule:

$$\frac{d}{dx}[x^n] = nx^{n-1}, \quad \frac{d}{dx}[x] = 1$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \quad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \quad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Chain Rule

Chain Rule:

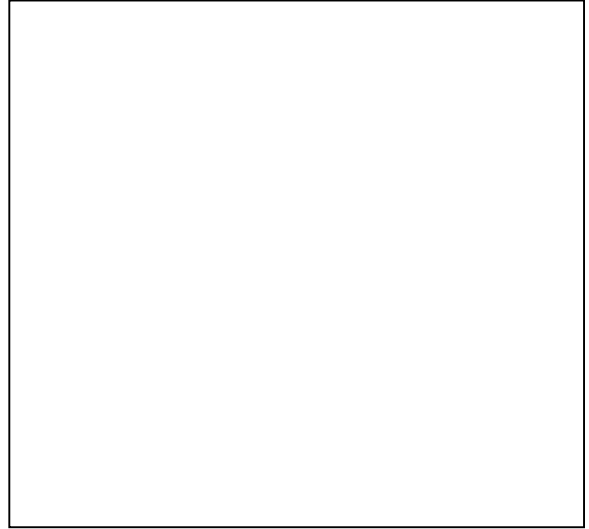
$$\frac{d}{dx}[f(u)] = f'(u) u'$$

General Power Rule:

$$\frac{d}{dx}[u^n] = nu^{n-1} u'$$

Ex.4 Find the derivative of $y = x^2\sqrt{16-x^2}$.

Ex.5 Find on the graph of $f(x) = \sqrt[3]{(x^2 - 1)^2}$ for which $f'(x) = 0$ and those for which $f'(x)$ does not exist.



Ex.6 Find the derivative of $y = \frac{t}{\sqrt{t^4 + 4}}$.

Ex.7 Find the derivative of $h(t) = \left(\frac{t^2}{t^3 + 2}\right)^2$.

Trigonometric Functions and the Chain Rule

The “Chain Rule versions” of the derivatives of the six trigonometric functions are as follows.

$$\frac{d}{dx}[\sin u] = (\cos u) u'$$

$$\frac{d}{dx}[\cos u] = -(\sin u) u'$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u) u'$$

$$\frac{d}{dx}[\cot u] = -(\csc^2 u) u'$$

$$\frac{d}{dx}[\sec u] = (\sec u \tan u) u'$$

$$\frac{d}{dx}[\csc u] = -(\csc u \cot u) u'$$

Ex.8 Applying the Chain Rule to Trigonometric Functions

a. $y = \sin \underbrace{2x}_u$ $y' = \cos 2x \underbrace{\frac{d}{dx}[2x]}_{u'} = (\cos 2x)(2) = 2 \cos 2x$

b. $y = \cos(x - 1)$ $y' = -\sin(x - 1)$

c. $y = \tan 3x$ $y' = 3 \sec^2 3x$

Ex.9 Derivatives, Parentheses, and Trigonometric Functions

Find the derivative of the following functions:

(a) $y = \cos 3x^2 = \cos(3x^2)$

(b) $y = (\cos 3)x^2$

(c) $y = \cos(3x)^2 = \cos(9x^2)$

(d) $y = \cos^2 x = (\cos x)^2$

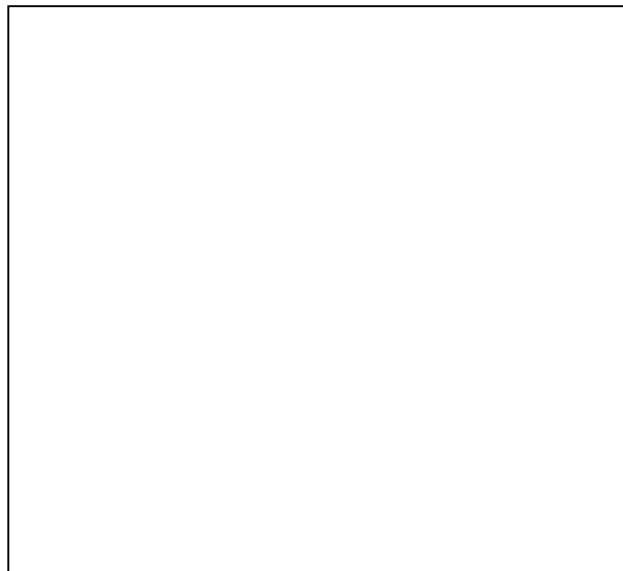
Ex.10 Find the derivative of $g(\theta) = 5\cos^2(\pi\theta)$.

Ex.11 Find the derivative of $g(\theta) = \cos\sqrt{\sin(\tan(\pi\theta))}$.

Ex.12 Evaluate the second derivative of $g(\theta) = \tan(2\theta)$ at $\left(\frac{\pi}{6}, \sqrt{3}\right)$.



Ex.13 Find the equation of the tangent line to the graph of $f(x) = 2\sin(x) + \cos(2x)$ at $(\pi, 1)$.
Then, find all values of x in $(0, 2\pi)$ at which the graph of f has a horizontal tangent.



Ex.14 Given $h(x) = f(g(x))$ and $s(x) = g(f(x))$, use the graphs of f and g to find the following derivatives:

(a) Find $h'(3)$.

(b) Find $s'(9)$.

